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RECOVERING A FRAME FROM ITS SHEAVES OF ALGEBRAS

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Let $\mathbb{H}$ be a frame, α an element of $\mathbb{H}$ and $\mathbf{T}$ a finitary algebraic theory. In this paper we compare the category $\mathrm{SH}(\mathbb{H},\mathbf{T})$ of sheaves of $\mathbf{T}$-algebras on $\mathbb{H}$ with the category $\mathrm{Sh}(\alpha\downarrow,\mathbf{T})$ of sheaves of $\mathbf{T}$-algebras on $\alpha\downarrow$ (where $\alpha\downarrow$ is the initial segment $\{\beta \mid \beta<\alpha\}$). This comparison suggests the definition of formal initial segments of the category $\mathrm{Sh}(\mathbb{H},\mathbf{T})$. For a large class of theories to be called 'integral' (examples of which are sets, monoids, groups, rings, modules on a integral domain, boolean algebras,...) the formal initial segments of $\mathrm{Sh}(\mathbb{H},\mathbf{T})$ coincide with the usual initial segments of $\mathbb{H}$: this means that we are able to recover $\mathbb{H}$ axiomatically from $\mathrm{Sh}(\mathbb{H},\mathbf{T})$.

When $\mathbb{H}$ is the initial frame $\{0,1\}$, the frame of formal initial segments of $\mathrm{Sh}(\mathbb{H},\mathbf{T})$ is the frame of open subsets of a compact space $\mathrm{Spec}\,\mathbf{T}$, called the spectrum of the theory $\mathbf{T}$. When $\mathbf{T}$ is the theory of modules on some ring R, we recover in this way various well known notions of spectra and their corresponding sheaf-representation of the ring.

Introduction

Throughout this paper, $\mathbb{H}$ denotes a frame (=local lattice) and $\mathbf{T}$ is a finitary algebraic theory; $\mathrm{Sh}(\mathbb{H},\mathbf{T})$ and $\mathrm{Pr}(\mathbb{H},\mathbf{T})$ denote the cateogry of sheaves and presheaves of $\mathbf{T}$-algebras on $\mathbb{H}$. If α is some element in $\mathbb{H}$, $\alpha\downarrow=\{\beta \mid \beta\leq\alpha\}$ is the corresponding initial segment and $\mathrm{Sh}\{\alpha\downarrow,\mathbf{T})$, $\mathrm{Pr}(\alpha\downarrow,\mathbf{T})$ the categories of sheaves and presheaves of $\mathbf{T}$-algebras on the frame $\alpha\downarrow$. $\mathrm{Sh}(\mathbb{H})$ and $\mathrm{Pr}(\mathbb{H})$ are the categories of sheaves and presheaves of sets on $\mathbb{H}$; $F:\mathrm{Pr}(\mathbb{H})\to\mathrm{Pr}(\mathbb{H},\mathbf{T})$ is the free-algebraic-presheaf functor and $a:\mathrm{Pr}(\mathbb{H})\to\mathrm{Sh}(\mathbb{H})$ the associated sheaf functor. In this way $aF:\mathrm{Sh}(\mathbb{H})\to\mathrm{Sh}(\mathbb{H},\mathbf{T})$ is a left adjoint to the forgetful functor (cf. [1]).

In a topos, the subobjects of any fixed object form a Heyting algebra. In $\mathrm{Sh}(\mathbb{H},\mathbf{T})$, the subobjects of any fixed object which satisfy distributivity conditions with respect to the intersection and union again form a Heyting algebra: we call these the Heyting subobjects; they play an important role when comparing $\mathrm{Sh}(\alpha\downarrow,\mathbf{T})$ and $\mathrm{Sh}(\mathbb{H},\mathbf{T})$. This comparison leads us to define the formal initial segments of $\mathrm{Sh}(\mathbb{H},\mathbf{T})$; these are the categories which satisfy axiomatically the basic properties of $\mathrm{Sh}(\alpha\downarrow,\mathbf{T})$ with respect to $\mathrm{Sh}(\mathbb{H},\mathbf{T})$ and they form canonically a frame $\mathscr{H}$ containing $\mathbb{H}$ as a subframe.

We introduce the notion of an integral theory; examples are given by sets, monoids, groups, rings, boolean algebras, modules on an integral domain,... When

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T is an integral theory, the frames $\mathbb{H}$ and $\mathscr{H}$ coincide; the frame $\mathbb{H}$ is thus completely characterized by its sheaves of **T**-algebras. When **T** is not an integral theory, we exhibit some relations between Sh($\mathbb{H}$, **T**) and Sh($\mathscr{H}$, **T**).

Now fix $\mathbb{H}$ to be the initial frame $\{0,1\}$. In this case $\mathscr{H}$, the frame of formal initial segments, is the frame of open subsets of a compact space Spec **T** which depends only on the theory **T**. This space Spec **T** is called the spectrum of the theory **T**. When **T** is the theory of modules on a ring R, Spec **T** coincides with the spectrum defined by Simmons (cf. [14]); When R is Von Neumann regular, Spec **T** is the Pierce spectrum of the ring (cf. [12]); when R is a Gelfand ring, Spec **T** is its maximal spectrum (cf. [2], [3], [10]); in general, the formal initial segments correspond to the pure ideals of the ring R (cf. [7], [14]).

Details of the proofs can be found in [4].

1. Basic properties of Sh($\mathbb{H}$, T)

If α is some element in $\mathbb{H}$, we denote by $h_\alpha : \mathbb{H}^{op} \to \text{Sets}$ the corresponding representable sheaf.

Proposition 1. *The categories* Pr($\mathbb{H}$, **T**) *and* Sh($\mathbb{H}$, **T**) *are complete, cocomplete and regular.* □

Proposition 2. *The presheaves* Fh_α $(\alpha \in \mathbb{H})$ *form a set of finitely presentable regular generators of* Pr($\mathbb{H}$, **T**).

Proof. By [6] 1-7 and 1-10 and the fact that in Pr($\mathbb{H}$, **T**) colimits are computed pointwise. □

Proposition 3. *The sheaves* $a\mathrm{Fh}_\alpha$ $(\alpha \in \mathbb{H})$ *form a set of regular generators of* Sh($\mathbb{H}$, **T**).

Proof. For Sh($\mathbb{H}$, **T**) is a localization of Pr($\mathbb{H}$, **T**). On the other hand colimits are no longer computed pointwise in Sh($\mathbb{H}$, **T**) and therefore $a\mathrm{Fh}_\alpha$ is not finitely presentable. □

Proposition 4. Sh($\mathbb{H}$, **T**) *possesses a dense family of generators such that for each object D of the family the canonical morphism* $0 \to D$ *is monic.*

Proof. By [6] 7-5, it suffices to consider the finite sums of sheaves $a\mathrm{Fh}_\alpha$ $(\alpha \in \mathbb{H})$. □

Proposition 5. *In* Pr($\mathbb{H}$, **T**) *and* Sh($\mathbb{H}$, **T**), *intersection commutes with filtered unions.*

Proof. By [13] 18-3-7 and the exactness of the associated sheaf functor. □

Proposition 6. *Let* $(A_i \rightarrowtail A)_{i\in I}$ *be a filtered family of subobjects in* Sh$(X, \mathbf{T})$ *and* $(f_i : A_i \to B)_{i\in I}$ *a family of morphisms such that, for* i, j *in* I*, the following square commutes:*

$$\begin{array}{ccc} A_i \cap A_j & \hookrightarrow & A_i \\ \cap\downarrow & & \downarrow f_i \\ A_j & \xrightarrow[f_i]{} & B \end{array}$$

There exists a unique morphism $f : \bigcup A_i \to B$ *extending the* f_i*'s. Moreover, if each* f_i *is monic, the same holds for* f.

Proof. Analogous to that of Proposition 5. □

2. The frame of Heyting subobjects

In a topos, the intersection with a subobject commutes always with the union of subobjects. For an algebraic category, this property is very rare: the main examples are the two trivial subobjects (the subobject of constants and the whole object); it's however possible to find non trivial examples (see Section 7 - pure ideals of a ring). In Pr$(\mathbb{H}, \mathbf{T})$, many non trivial examples of such subobjects $S \rightarrowtail A$ can be found by choosing $P(\alpha)$ to be trivial for each α in $\mathbb{H}$; passing to the associated sheaves, one gets examples in Sh$(\mathbb{H}, \mathbf{T})$. These last examples are clearly related to the structure of $\mathbb{H}$ and thus provide a first step to a characterization of $\mathbb{H}$ from Sh$(\mathbb{H}, \mathbf{T})$.

Definition 7. Let $\mathbb{C}$ be a finitely complete and cocomplete regular category. A subobject $S \rightarrowtail A$ in $\mathbb{C}$ is called a Heyting subobject if for any subobjects R, T of A the following conditions are satisfied.

(1) $S \cap (R \cup T) \cong (S \cap R) \cup (S \cap T)$.

(2) $R \cap (S \cup T) \cong (R \cap S) \cup (R \cap T)$.

(3) The square

$$\begin{array}{ccc} S \cap R \cap T & \longrightarrow & R \cap T \\ \downarrow & & \downarrow \\ S \cap T & \longrightarrow & (S \cap T) \cup (R \cap T) \end{array}$$

is cocartesian.

Proposition 8. *Let* $S \rightarrowtail A$, $(S_i \rightarrowtail A)_{i\in I}$ *be Heyting subobjects in* Sh(ℍ, **T**). *For any* subobjects $R \rightarrowtail A$, $T \rightarrowtail A$, $(R_i \rightarrowtail A)_{i\in I}$ *the following relations hold.*

(i) $S\cup(R\cap T)\cong(S\cup R)\cap(S\cup T)$.

(ii) $R\cup(S\cap T)\cong(R\cup S)\cap(R\cup T)$.

(iii) *There exists a subobject* $S\Rightarrow R$ *of* A *such that*

$$T\leq S\Rightarrow R \quad \textit{iff} \quad T\cap S\leq R.$$

(iv) $S\cap(\bigcup_{i\in I} R_i)\cong\bigcup_{i\in I}(S\cap R_i)$.

(v) $R\cap((\bigcup_{i\in I} S_i)\cup T)\cong\bigcup_{i\in I}(R\cap S_i)\cup(R\cap T)$.

(vi) $S\cap R$ *is a Heyting subobject of* R.

Proof. Replace $\bigcup_{i\in I}$ by the filtered union $\bigcup_{J\in I}$ where J runs through the finite subsets of I; (iv) and (v) follow then from Proposition (5) and (i), (ii), (iii) from the general theory of complete distributive lattices (cf. [18]); the proof of (vi) is quite straightforward (using (i) and (ii)). □

Theorem 9. *For any object* A *in* Sh(ℍ, **T**), *the Heyting subobjects of* A *are an* $\bigvee\bigwedge$*-sublattice and are a frame.*

Proof. Let $S\rightarrowtail A$, $S'\rightarrowtail A$ and $(S_i\rightarrowtail A)_{i\in I}$ be Heyting subobjects of A.

$S\cap S'$ satisfies (1) and (2) in Definition 7 (Proposition 8). To prove (3), consider the cocartesian square with T replaced by $T\cap S'$. Then consider $(S\cap T)\cup(R\cap T)$ with its Heyting subobject given by $S'\cap((S\cap T)\cup(R\cap T))=(S\cap S'\cap T)\cup(S'\cap T\cap R)$.

$\bigcup_{i\in I} S_i$ satisfies (1) and (2) in Definition 7 (Proposition 8). To prove (3), consider first S_i, R and T as subobjects of A, then S_i, $(R\cup S_i)$ and T, and so on, ...; conclude by Proposition 6. □

3. The frame of formal initial segments

For each α in ℍ, we prove that Sh($\alpha\downarrow$, **T**) is a localization of Sh(ℍ, **T**) satisfying various additional properties; some of them can be expressed in terms of Heyting subobjects. Any localization of Sh(ℍ, **T**) which satisfies these properties is called a formal initial segment. These formal initial segments form a frame containing ℍ as a subframe and contained itself as a subframe in the frame of Heyting subobjects of $a\mathrm{Fh}_1$.

Theorem 10. *Let* α *be some element in* ℍ.

(1) *The restriction functor* α^*: Sh(ℍ, **T**) → Sh($\alpha\downarrow$, **T**) *has a full and faithful left adjoint* $\alpha_!$ *and a full and faithful right adjoint* α_*.

(2) $\alpha_!$ *preserves and creates monomorphisms.*

(3) *If* $0\rightarrow A$ *is monic in* Sh(ℍ, **T**), *the canonical morphism* $\alpha_!\alpha^*A\rightarrow A$ *is a Heyting subobject.*

Proof. Let A be some object in Sh($\alpha\downarrow$, **T**) and β some element in $\mathbb{H}$. Define α_* by $\alpha_*(A)(\beta) = A(\alpha\wedge\beta)$. Define $\alpha_!(A)$ as the sheaf associated to the presheaf $\alpha'(A)$ given by

$$\alpha'(A)(\beta) = \begin{cases} A(\beta) & \text{if } \beta\le\alpha, \\ 0 & \text{if } \beta\not\le\alpha, \end{cases}$$

Clearly $\alpha_! \dashv \alpha^* \dashv \alpha_*$ and α_* is full and faithful; therefore $\alpha_!$ is also full and faithful (cf. [13] 16-8-9).

$\alpha_!$ preserves and creates monomorphisms by construction and the exactness of the associated sheaf functor.

$\alpha'\alpha^*A$ is a subobject of A in Pr($\mathbb{H}$, **T**) and thus so is $\alpha_!\alpha^*A$ in Sh($\mathbb{H}$, **T**). In each component $\alpha'\alpha^*A(\beta)$ is a trivial subobject of $A(\beta)$; therefore $\alpha'\alpha^*A$ is a Heyting subobject of A in Pr($\mathbb{H}$, **T**). The properties of the associated sheaf functor imply that $\alpha_!\alpha^*A$ is a Heyting subobject of A in Sh($\mathbb{H}$, **T**). □

Definition 11. Let $\mathbb{C}$ be a finitely complete and cocomplete regular category. A formal initial segment α of $\mathbb{C}$ is a full subcategory α such that

(1) the canonical inclusion $\alpha_! : \alpha\to\mathbb{C}$ has a right adjoint α^* which has itself a right adjoint α_*.

(2) If $0\to A$ and $S\to A$ are monic in $\mathbb{C}$ and if A is an object in α, S is also in α.

(3) If $0\to A$ is monic in $\mathbb{C}$, the canonical morphism $\alpha_!\alpha^*A\to A$ is a Heyting subobject.

Proposition 12. *Let $\mathbb{C}$ be a finitely complete and cocomplete regular category and α a formal initial segment of $\mathbb{C}$.*

(1) *If $0\to A$ is monic in $\mathbb{C}$, $\alpha_!$ preserves the monomorphisms with codomain A in α.*

(2) *$\alpha_!$ reflects monomorphisms.*

(3) *α_* is full and faithful.*

(4) *If $0\to A$ and $S\to A$ are monic in $\mathbb{C}$, $\alpha_!\alpha^*S = S\cap\alpha_!\alpha^*A$ as subobjects of A.*

(5) *α is saturated under isomorphisms.*

Proof. (1) Factor a monomorphism in α through its image in $\mathbb{C}$ and use the exactness of α^*.

(2) α_* is full and faithful (cf. [13]).

(3) $\alpha_!$ is full and faithful (cf. [13] 16-8-9).

(4) The canonical monomorphism $\alpha_!\alpha^*S\to A\wedge\alpha_!\alpha^*A$ has a right inverse $\alpha_!\alpha^*(S\cap\alpha_!\alpha^*A\rightarrowtail S)$ and thus is an isomorphism.

(5) Condition 2 in Definition 11.

Theorem 13. *The formal initial segments of* Sh($\mathbb{H}$, **T**) *are a frame* $\mathscr{H}$.

Proof. The formal initial segments are ordered by inclusion of subcategories.

Sh(ℍ, **T**) is the greatest element and $\{0\}$ with $\alpha_*(0)=1$ and $\alpha_!(0)=0$ is the smallest element.

Let α, β be two formal initial segments of Sh(ℍ, **T**). Define their intersection $\alpha\wedge\beta$ as the full subcategory $\alpha\cap\beta$, $(\alpha\wedge\beta)_!$ is thus the canonical inclusion. Now the two functors $\alpha_!\alpha^*\beta_!\beta^*$ and $\beta_!\beta^*\alpha_!\alpha^*$ are right exact and coincide on a dense subcategory of Sh(ℍ, **T**) (Proposition 4 and 12(4)); from [13] 17-2-7 and 17-2-8 one deduces that these two functors are naturally isomorphic: this is $(\alpha\wedge\beta)^*$. From that $\alpha_*\alpha^*\beta_*\beta^*$ and $\beta_*\beta^*\alpha_*\alpha^*$ are naturally isomorphic and equal to $(\alpha\wedge\beta)_*$. It's routine to check that $\alpha\wedge\beta$ is a formal initial segment.

Let $(\alpha_i)_{i\in I}$ be a family of formal initial segments of Sh(ℍ, **T**), we need to define $\bigvee_{i\in I}\alpha_i$. We define first a functor $G:\mathrm{Sh}(\mathbb{H},\mathbf{T})\to\mathrm{Sh}(\mathbb{H},\mathbf{T})$ by taking $G(A)$ to be the colimits of the diagram

$$(\alpha_i\wedge\alpha_j)_!(\alpha_i\wedge\alpha_j)^*(A)\ \begin{matrix}\nearrow & \alpha_{i!}\alpha_i^*(A)\\ \searrow & \alpha_{j!}\alpha_j^*(A)\end{matrix}$$

for any indexes i, j. We define $\bigvee_{i\in I}\alpha_i$ as the full subcategory of Sh(ℍ, **T**) spanned by all the objects $G(A)$ and saturated for the isomorphisms. Let us prove that $\bigvee_{i\in I}\alpha_i$ is a formal initial segment.

G factors through $\bigvee_{i\in I}\alpha_i$ into $(\bigvee_{i\in I}\alpha_i)_!(\bigvee_{i\in I}\alpha_i)^*$. Now for A in Sh(ℍ, **T**) the canonical morphism $\theta_i:\alpha_{i!}\alpha_i^*A\to A$ factors through $(\bigvee_{i\in I}\alpha_i)_!(\bigvee_{i\in I}\alpha_i)^*(A)$; this gives rise to a morphism

$$\theta_A:\Big(\bigvee_{i\in I}\alpha_i\Big)_!\Big(\bigvee_{i\in I}\alpha_i\Big)^*(A)\to(A).$$

Moreover for any i in I

$$(\alpha_i)_!(\alpha_i)^*\Big(\bigvee_{i\in I}\alpha_i\Big)_!\Big(\bigvee_{i\in I}\alpha_i\Big)^*\cong(\alpha_i)_!(\alpha_i)^*$$

and from this one deduces that θ_A has the universal property presenting $(\bigvee_{i\in I}\alpha_i)^*$ as a right adjoint to $(\bigvee_{i\in I}\alpha_i)_!$.

Next one shows that the image of θ_A is the union of the images of the $(\theta_i)_A$ and from Propositions 5, 6 and 10 one deduces the required properties of $(\bigvee_{i\in I}\alpha_i)_!$.

Finally, for any A in Sh(ℍ, **T**) we define $(\bigvee_{i\in I}\alpha_i)_*(A)$ as the limit of the diagram

$$\begin{matrix}(\alpha_i)_*(\alpha_i)^*\Big(\bigvee_{i\in I}\alpha_i\Big)_!(A) & \searrow & \\ & & (\alpha_i\wedge\alpha_j)_*(\alpha_i\wedge\alpha_j)^*\Big(\bigvee_{i\in I}\alpha_i\Big)_!(A)\\ (\alpha_j)_*(\alpha_j)^*\Big(\bigvee_{i\in I}\alpha_i\Big)_!(A) & \nearrow & \end{matrix}$$

and we get the right adjoint to $(\bigvee_{i\in I}\alpha_i)^*$.

In order to have a frame, all that remains is to prove the distributivity laws between $\bigvee$ and $\wedge$.

It is sufficient to prove that if α and $(\beta_i)_{i\in I}$ are formal initial segments of Sh(IH, **T**),

$$\left[\alpha\wedge\left(\bigvee_{i\in I}\beta_i\right)\right]_!\left[\alpha\wedge\left(\bigvee_{i\in I}\beta_i\right)\right]^*\cong\left[\bigvee_{i\in I}(\alpha\wedge\beta_i)\right]_!\left[\bigvee_{i\in I}(\alpha\wedge\beta_i)\right]^*.$$

But these functors are right exact and so it suffices to prove the isomorphism on a dense subcategory. By Proposition 4, this follows from the definition of Heyting subobjects and formal initial segments. □

Proposition 14. *Let* $\mathscr{H}$ *be the frame of formal initial segments of* Sh(IH, **T**) *and* Heyt(aFh$_1$) *the frame of Heyting subobjects of* aFh$_1$. *There are inclusions of frames*

$$\mathbb{H}\subseteq\mathscr{H}\subseteq \text{Heyt}(a\text{Fh}_1)$$

Proof. Consider the application $\mathscr{H}\to$Heyt(aFh$_1$); $\alpha\to\alpha_!\alpha^*(a\text{Fh}_1)$. It is injective by Propositions 4 and 12(4); it is an inclusion of frames by construction of $\mathscr{H}$. The inclusion $H\subseteq\mathscr{H}$ follows from Theorem 10. □

4. Integral theories and the characterization theorem

The characterization theorem we have in view gives a condition under which a frame IH is completely characterized by its category Sh(IH, **T**) of sheaves of algebras for an algebraic theory **T**. The condition is on **T**; with this condition, the frame IH is nothing but the frame of formal initial segments of Sh(IH, **T**). We give also a counterexample to prove that the characterization theorem does not hold for an arbitrary theory **T**.

Definition 15. A finitary algebraic theory **T** is called integral if any non-constant 1-ary operation is an epimorphism in the category **T**.

Proposition 16. *A finitary algebraic theory* **T** *is integral if and only if for any non-constant element* x *of the free algebra* $F1$ *on one generator, the canonically induced morphism* $\hat{x}:F1\to F1$ *is injective.* □

Proposition 17. *Let* A *be a ring. The theory of* A *modules is integral if and only if the ring* A *is an integral domain.* □

Proposition 16 justifies the terminology of Definition 15. The examples of integral theories are numerous as Proposition 17 shows.

Proposition 18. *The following theories are integral: sets with or without base point(s); sets on which a groups acts; monoids; abelian groups; rings with or without unit, commutative or not; modules on an integral domain; vector spaces on a field; boolean algebras.* □

Theorem 19 (the characterization theorem). *Let* **T** *be an integral theory. Two frames* $\mathbb{H}$ *and* $\mathbb{H}'$ *are isomorphic if and only if the corresponding categories* $\mathrm{Sh}(\mathbb{H},\mathbf{T})$ *and* $\mathrm{Sh}(\mathbb{H},\mathbf{T}')$ *of sheaves of* **T**-*algebras are equivalent.*

Proof. We simply prove that if **T** is integral, $\mathbb{H}$ is isomorphic to the frame $\mathscr{H}$ of formal initial segments of $\mathrm{Sh}(\mathbb{H},\mathbf{T})$. By Proposition 14, it remains to show that each formal initial segment α of $\mathrm{Sh}(\mathbb{H},\mathbf{T})$ arises from an element β of $\mathbb{H}$. Again from Proposition 14 it suffices to prove that $\alpha_!\alpha^*(a\mathrm{Fh}_1)$ has the form $a\mathrm{Fh}_\beta$.

By Proposition 14, let β be the greatest element of $\mathbb{H}$ smaller than α in $\mathscr{H}$. For any $\gamma\in\mathbb{H}$ denote by Sh_γ the separated presheaf of **T**-algebras universally associated to h_γ and by P the intersection of Sh_1 and $\alpha_!\alpha^*(a\mathrm{Fh}_1)$ as subobjects of $a\mathrm{Fh}_1$. $\alpha_!\alpha^*(a\mathrm{Fh}_1)$ is the sheaf universally associated to P and Sh_β is a subobject of P.

If $\alpha_!\alpha^*(a\mathrm{Fh}_1)$ is not isomorphic to $a\mathrm{Fh}_\beta\cong a\mathrm{Sh}_\beta$, there exists some γ in $\mathbb{H}$, $\gamma\not\leq\beta$, such that $P(\gamma)\neq\mathrm{Sh}_\beta(\gamma)$. But $\mathrm{Sh}_\beta(\gamma)=F0$; so we can find $x\in P(\gamma)$ which is not a constant. Denote by $\langle x\rangle$ the sub-presheaf of **T**-algebras of P generated by x. There is a canonical surjection of presheaves $\mathrm{Sh}_\gamma\to\langle x\rangle$ which is an isomorphism because **T** is integral. Therefore $\alpha_!\alpha^*(a\mathrm{Sh}_\gamma)$ is isomorphic to $a\mathrm{Sh}_\gamma$ and thus γ is smaller than α: finally $\gamma\leq\beta$, which is a contradiction. □

Finally we give a counterexample to the characterization theorem in the case of a non-integral theory. Let $\mathbb{N}$ be the set of integers and $\mathscr{P}(\mathbb{N})$ the boolean ring of subsets of $\mathbb{N}$. $\mathbb{N}$ is isomorphic to $\mathbb{N}\amalg\mathbb{N}$ and thus there is an isomorphism of rings $\mathscr{P}(\mathbb{N})\cong\mathscr{P}(\mathbb{N}\amalg\mathbb{N})\cong\mathscr{P}(\mathbb{N})\times\mathscr{P}(\mathbb{N})$. Therefore we have an isomorphism between the categories of modules (cf. [11])

$$\mathrm{Mod}_{\mathscr{P}(\mathbb{N})}\cong\mathrm{Mod}_{\mathscr{P}(\mathbb{N})}\times\mathrm{Mod}_{\mathscr{P}(\mathbb{N})}$$

This shows that the categories of sheaves of $\mathscr{P}(\mathbb{N})$-modules on the spaces 1 and 2 are equivalent. Thus a frame cannot be characterized by its sheaves of $\mathscr{P}(\mathbb{N})$-modules.

5. Sheaves on the frame of formal initial segments

In general, the frame $\mathscr{H}$ of formal initial segments of $\mathrm{Sh}(\mathbb{H},\mathbf{T})$ is not isomorphic to $\mathbb{H}$. In this section, we compare the categories $\mathrm{Sh}(\mathbb{H},\mathbf{T})$ and $\mathrm{Sh}(\mathscr{H},\mathbf{T})$. They are clearly related by the restriction functor

$$R:\mathrm{Sh}(\mathscr{H},T)\to\mathrm{Sh}(\mathbb{H},T).$$

Proposition 20. *With the notations above, R has a left exact left adjoint T.*

Proof. For A in $\mathrm{Sh}(\mathbb{H},\mathbf{T})$, define TA as the sheaf associated to the presheaf $T'A$ given by

$$T'A(\alpha)=\varinjlim_{\substack{\beta\geq\alpha\\ \beta\in\mathbb{H}}} A(\beta).$$

T is left exact because in $\mathrm{Sets}^{\mathbf{T}}$ finite limits commute with filtered colimits (cf. [13]). □

We shall now define another functor $S:\mathrm{Sh}(\mathbb{H},\mathbf{T})\to\mathrm{Sh}(\mathscr{H},\mathbf{T})$ which is not adjoint to R but has the interesting property that $R\circ S\cong\mathrm{id}$. If A is some object in $\mathrm{Sh}(\mathbb{H},\mathbf{T})$ and $\alpha\in\mathscr{H}$, we define

$$SA(\alpha)=(\alpha_*\alpha^*A)(1).$$

Proposition 21. *With the notations above, SA becomes a sheaf and S extends into a faithful and limit-preserving functor $S:\mathrm{Sh}(\mathbb{H},\mathbf{T})\to\mathrm{Sh}(\mathscr{H},\mathbf{T})$ such that $R\circ S\cong\mathrm{id}$.* □

Proposition 22. *$S(a\mathrm{Fh}_1)$ is a monoid in the topos $\mathrm{Sh}(\mathscr{H})$ of sheaves on $\mathscr{H}$.*

Proof. For any α in $\mathscr{H}$,

$$S(a\mathrm{Fh}_1)(\alpha)\cong\mathrm{Sh}(\alpha\downarrow,\mathbf{T})(\alpha^*a\mathrm{Fh}_1,\alpha^*a\mathrm{Fh}_1).$$

and the structure of monoid is given by composition. □

Proposition 23. *For any sheaf $A\in\mathrm{Sh}(\mathbb{H},\mathbf{T})$, the sheaf SA is provided with an action*

$$S(a\mathrm{Fh}_1)\times SA\to SA$$

of the monoid $S(a\mathrm{Fh}_1)$.

Proof. The action is again given by composition. □

6. The spectrum of an algebraic theory

The frame $\mathscr{H}$ of formal initial segments of $\mathrm{Sh}(\mathbb{H},\mathbf{T})$ depends on $\mathbb{H}$ and on $\mathbf{T}$. Now if we fix $\mathbb{H}$ to be the initial frame $\{0,1\}$, we get a frame $\mathscr{H}$ which depends only on $\mathbf{T}$ and which can be non-trivial if $\mathbf{T}$ is not integral. In this case we show that $\mathscr{H}$ is in fact the algebra of open subsets of some compact space X which we call the spectrum of the theory $\mathbf{T}$; examples are given in the next paragraph. Moreover, the functor S of Section 5 presents any $\mathbf{T}$-algebra as the algebra of global sections of a sheaf of $\mathbf{T}$-algebras on the spectrum of $\mathbf{T}$.

Definition 24. A $\mathbf{T}$-ideal is a subalgebra of the free algebra $F1$ on a single generator.

Definition 25. A $\mathbf{T}$-ideal is called pure if it is of the form $\alpha_! \alpha^*(F1)$ for some formal initial segment α of sets$^{\mathbf{T}}$.

Definition 26. A pure $\mathbf{T}$-ideal J is called purely prime if it is proper and if for any pure $\mathbf{T}$-ideals I_1, I_2

$$I_1 \cap I_2 \subseteq J \ \Rightarrow \ I_1 \subseteq J \text{ or } I_2 \subseteq J.$$

Definition 27. A pure $\mathbf{T}$-ideal J is called purely maximal if it is maximal among the proper pure $\mathbf{T}$-ideals.

Proposition 28. *The pure* $\mathbf{T}$*-ideals form a subframe* $\mathscr{W}$ *of the lattice of subalgebras of* F_1.

Proof. By Theorem 13 and Proposition 14. □

Proposition 29. *Any purely maximal* $\mathbf{T}$*-ideal is purely prime.* □

Proposition 30. *Any proper pure* $\mathbf{T}$*-ideal is contained in a purely maximal* $\mathbf{T}$*-ideal.*

Proof. By Zorn's lemma. □

Proposition 31. *Let I be a pure* $\mathbf{T}$*-ideal and $a \in F1 \setminus I$. There exists a purely prime ideal J such that $I \subseteq J$ and $a \notin J$.*

Proof. By Zorn's lemma and Proposition 28. □

Proposition 32. *Any proper pure* $\mathbf{T}$*-ideal is the intersection of the purely prime* $\mathbf{T}$*-ideals containing it.*

Proof. By Proposition 31. □

We turn now to the construction of the (purely prime) spectrum $S_{pp}(\mathbf{T})$ of the theory $\mathbf{T}$. Its underlying set is nothing but the set of all purely prime $\mathbf{T}$-ideals. if $\mathscr{P}(S_{pp}\mathbf{T})$ denotes the power set of $S_{pp}(\mathbf{T})$ and $\mathscr{W}$ the frame of pure $\mathbf{T}$-ideals, we can define a map

$$\mathscr{O} : \mathscr{W} \to \mathscr{P}(S_{pp}\mathbf{T})$$

by $\mathscr{O}(I) = \{J \mid J \in S_{pp}(\mathbf{T});\ I \not\subseteq J\}$.

Proposition 33. $\mathscr{O}$ *is an injection of frames.* □

Proposition 34. *The subsets $\mathcal{O}(I)$ of $S_{\mathrm{pp}}(T)$, when I runs through $\mathcal{H}$, form a topology on $S_{\mathrm{pp}}(T)$.*

Proof. By Proposition 33. □

Definition 35. The (purely prime) spectrum of an algebraic theory **T** is the set $S_{\mathrm{pp}}(\mathbf{T})$ provided with the topology arising from Proposition 34.

$S_{\mathrm{pp}}(\mathbf{T})$ is clearly the space universally associated to the frame of pure **T**-ideals. But we have proved more: the frame of pure **T**-ideals is exactly the frame of open subsets of $S_{\mathrm{pp}}(\mathbf{T})$. (This is a particular case of Buchi's classical result).

Proposition 36. *The (purely prime) spectrum of an algebriac theory* **T** *is a compact topological space.*

Proof. If $(I_k)_{k\in K}$ is a family of pure **T**-ideals such that $\bigcup_{k\in K} I_k = F1$, the canonical generator of $F1$ can be expressed as an algebraic combination of elements of a finite number of elements choosen in a finite number of $I_{k_1},\dots,I_{k_n}$. Therefore $F1 \cong \bigcup_{i=1}^{n} I_{k_i}$. □

Proposition 37. *Let S be the functor described in Section 5. For any* **T***-algebra A, SA is a sheaf of* **T***-algebras on* $S_{\mathrm{pp}}(\mathbf{T})$ *whose algebra of global sections is exactly A.*

Proof. By Proposition 21 and the fact that in the particular context of this paragraph the functor R of Section 5 is exactly the global sections functor. □

7. Applications to the theory of rings

In this paragraph we consider the theory **T** of left R-modules where R is an arbitrary ring with a unit. A **T**-ideal is just an ordinary left ideal of R. We exhibit the form of the pure **T**-ideals and get what is called in the literature 'pure ideals' (cf. [2], [3], [5], [10]) or 'still ideals' (cf. [14]). This gives an easy description of the purely prime spectrum of **T** which we simply call the purely prime spectrum of the ring R. This spectrum coincides with the one described by Simmons (cf. [14]) in the case of a general ring, with Pierce's spectrum (cf. [12]) in the case of a commutative von Neumann regular ring, with the spectrum described by Bkouche and Mulvey (cf. [2], [3], [10]) in the case of a commutative Gelfand ring. Moreover Propositions 21, 22, 23, 37 provide a sheaf representation of the ring R and of any R-module; this representation is again the one described by Simmons, Pierce, Bkouche, Mulvey in the cases we have already mentioned. We prove that the rings of local sections of this representation are exactly the rings of endomorphisms of the pure ideals of R; this gives an easy and unified sheaf description of the various corresponding 'espaces étalés'.

In this section, we make explicit the relation between the formal initial segments of the category of R-modules and the pure ideals of R. The subsequent results on the representation of rings are simply mentioned; they will be the object of a further publication on ring theory.

Proposition 38. *Let α be a formal initial segment of the category* Mod_R *of left R-modules. if $I=\alpha_!\alpha^*R$, α is the full subcategory of those R-modules M such that $IM=M$ and $\alpha^*M=IM$.*

Proof. $\alpha_!\alpha^*$ preserves colimits and each R-module can be expressed as a colimit of a diagram build from the single object R. □

Proposition 39. *Let α be a formal initial segment of* Mod_R *and $I=\alpha_!\alpha^*R$. I is a pure ideal, i.e. a two-sided such that*

$$\forall i\in I\ \exists\varepsilon\in I\quad \varepsilon i=i.$$

Proof. If J is any left ideal of R,

$$IJ=\alpha_!\alpha^*J=\alpha_!\alpha^*R\cap J=I\cap J$$

by Propositions 12(4) and 38. In particular $IR=I$ and I is a two-sided ideal.

Now consider $i\in I$. The following equalities hold:

$$I\langle i\rangle=I\wedge\langle i\rangle=\langle i\rangle,$$

which show the existence of the required ε. □

In order to prove the converse of Proposition 39, we recall some characterizations of pure ideals.

Proposition 40. *The following conditions are equivalent for a two-sided ideal I of R*

(1) *I is a pure ideal.*

(2) $\forall i\in I\ \exists\varepsilon\in I\ \varepsilon i=i$.

(3) $\forall i_1,\dots,i_n\in I\ \exists\varepsilon\in I$ *such that* $\forall k\ \varepsilon i_k=i_k$.

(4) *For any left R-module M, $I\otimes M\cong IM$.*

(5) *R/I is a flat right R-module.*

Proof. See [14]. □

Theorem 41. *There is an isomorphism between the frame of pure ideals of the ring R and the frame of formal initial segments of the category* Mod_R.

Proof. It remains to be shown that each pure ideal I has the form $\alpha_!\alpha^*R$ for some formal initial segment α of the category Mod_R. Choose α to be the full subcategory of these modules M such that $I\otimes M\cong IM\cong M$; $\alpha^*M\cong I\otimes M$ and $\alpha_*N\cong\mathrm{Mod}_R(I;N)$. □

We denote by $S_{pp}R$ the purely prime spectrum of R, i.e. the purely prime spectrum of the theory of left R-modules. Its frame $\mathcal{O}(S_{pp}R)$ is thus exactly the frame of pure ideals of R.

Proposition 42. *The application* $\mathcal{O}(S_{pp}R)\to$Rings *which associates to the pure ideal* I *the ring* $\mathrm{End}_R(I)$ *of R-endomorphisms of I extends into a sheaf of rings.*

Proof. By Propositions 21, 22, 23 and Theorem 41 which show that if α is the formal initial segment associated to I, $\alpha_*(I)\cong\mathrm{End}_R(I)$. □

If R is a commutative Von Neumann regular ring, $S_{pp}(R)$ is the spectrum of R described by Pierce in [12] and proposition 42 gives the sheaf presentation of the 'espace étalé' described by Pierce. More generally, if R is commutative Gelfand ring (cf. [2], [3], [10]), $S_{pp}R$ is the maximal spectrum of R and Proposition 42 gives the sheaf presentation of the 'espaces étalés' described by Grothendieck, Bkouche and Mulvey.

Finally we introduce the notion of a pure ring; the relations with Gelfand rings will be studied in a further publication as well as the fundamental properties, of pure rings.

Definition 43. Let $\alpha\leq\beta$ be two elements in a lattice with greatest element, α is called dense in β if for any γ

$$\beta\vee\gamma=1 \;\Rightarrow\; \alpha\vee\gamma=1.$$

Definition 44. A ring R with a unit is called pure if for any ideal I of R there is a pure ideal J which is dense in I (in the sense of Definition 43).

Theorem 45. *Let R be a commutative pure ring. The purely prime spectrum of R is a compact Hausdorff space and the stalks of the sheaf representation of R given by Proposition* 42 *are local rings.* □

References

[1] M. Artin, A. Grothendieck and J.L. Verdier, Théorie des topos et cohomologie étale des schémas, Lecture Notes in Math. No. 269 (Springer, Berlin, 1972).

[2] R. Bkouche, Pureté, mollesse et paracompacité, C.R.Acad.Sci. Paris 270 (1970) 1653-1655.

[3] R. Bkouche, Espaces annelés commutatifs compacts, C.R.Acad.Sci. Paris A-273 (1971) 1200–1203.

[4] F. Borceux and G. Van den Bossche, Recovering a frame from its sheaves of algebras (detailed version), Rapp. Sém. Math. Pures, Université Louvain (1981).

[5] C. Faith, Ring Theory, Grund. der math. Wiss. Vol. 191 (Springer, Berlin, 1976).

[6] P. Gabriel and F. Ulmer, Lokal präsentierbare Kategorien, Lecture Notes in Math. No. 221 (Springer, Berlin, 1971).

[7] J. Golan, Localization of non-commutative rings (Marcel Dekker, New York, 1975).

[8] G. Gratzer, General Lattice Theory (Birkaüser, Basel, 1978).
[9] A. Grothendieck and J. Dieudonné, Eléments de géométrie algébrique, I, le langage des schémas, Inst. Hautes Etudes Sci. Publ. Math. No. 4 (1960).
[10] C.J. Mulvey, A generalization of Swan's theorem, Math. Z. 151 (1976) 57–70.
[11] C.J. Mulvey, Representations of rings and modules. Applications of sheaves, Lecture Notes in Math. No. 753 (Springer, Berlin, 1979) 542–582.
[12] R.S. Pierce, Modules over commutative regular rings, Mem. Amer. Math. Soc. 77 (1971) 386–391.
[13] H. Schubert, Categories (Springer Berlin–New York, 1972).
[14] H. Simmons, The frame of localizations of a ring, Preprint, Univ. Aberdeen (1981).